Chapter 6

Noise in Modulation Systems

6.1 Problems

Problem 6.1
The signal power at the output of the lowpass filter is $P_T$. The noise power is $N_0B_N$, where $B_N$ is the noise-equivalent bandwidth of the filter. From (5.116), we know that the noise-equivalent bandwidth of an $n^{th}$ order Butterworth filter with 3 dB bandwidth $W$ is

$$B_n(n) = \frac{\pi W/2n}{\sin(\pi/2n)}$$

Thus, the signal-to-noise ratio at the filter output is

$$SNR = \frac{P_T}{N_0B_N} = \frac{\sin(\pi/2n)}{\pi/2n} \frac{P_T}{N_0W}$$

so that $f(n)$ is given by

$$f(n) = \frac{\sin(\pi/2n)}{\pi/2n}$$

We can see from the form of $f(n)$ that since $\sin(x) \approx 1$ for $x \ll 1$, $f(\infty) = 1$. Thus for large $n$, the SNR at the filter output is close to $P_T/N_0W$. The plot is shown in Figure 6.1.

Problem 6.2
We express $n(t)$ as

$$n(t) = n_c(t)\cos\left[\omega_c t + \frac{1}{2} (2\pi W) t + \theta\right] + n_s(t)\sin\left[\omega_c t + \frac{1}{2} (2\pi W) t + \theta\right]$$

where we use the plus sign for the $USB$ and the minus sign for the $LSB$. The received signal is

$$x_r(t) = A_c \left[m(t)\cos(\omega_c t + \theta) \mp \tilde{m}(t)\sin(\omega_c t + \theta)\right]$$
Multiplying \( x_r(t) + n(t) \) by \( 2 \cos (\omega_c t + \theta) \) and lowpass filtering yields
\[
y_D(t) = A_c m(t) + n_c(t) \cos (\pi W t) + n_s(t) \sin (\pi W t)
\]
From this expression, we can show that the postdetection signal and noise powers are given by
\[
S_D = A_c^2 m^2 \quad N_D = N_0 W
\]
\[
S_T = A_c^2 m^2 \quad N_T = N_0 W
\]
This gives a detection gain of one. The power spectral densities of \( n_c(t) \) and \( n_s(t) \) are illustrated in Figure 6.2.

**Problem 6.3**
The received signal and noise are given by
\[
x_r(t) = A_c m(t) \cos (\omega_c t + \theta) + n(t)
\]
At the output of the predetection filter, the signal and noise powers are given by
\[
S_T = \frac{1}{2} A_c^2 m^2 \quad N_T = \bar{n}^2 = N_0 B_T
\]
The predetection SNR is
\[
(SNR)_T = \frac{A_c^2 m^2}{2 N_0 B_T}
\]
If the postdetection filter passes all of the \( n_c(t) \) component, \( y_D(t) \) is

\[
y_D(t) = A_c m(t) + n_c(t)
\]

The output signal power is \( A^2_c m^2 \) and the output noise PSD is shown in Figure 6.3.

Case I: \( B_D > \frac{1}{2} B_T \)

For this case, all of the noise, \( n_c(t) \), is passed by the postdetection filter, and the output noise power is

\[
N_D = \int_{-\frac{1}{2} B_D}^{\frac{1}{2} B_D} N_0 df = 2 N_0 B_T
\]

This yields the detection gain

\[
\frac{(SNR)_D}{(SNR)_T} = \frac{A^2_c m^2 / N_0 B_T}{A^2_c m^2 / 2 N_0 B_T} = 2
\]

**Figure 6.2:**

![Figure 6.2:](image)

**Figure 6.3:**

![Figure 6.3:](image)
Case II: \( B_D < \frac{1}{2}B_T \)

For this case, the postdetection filter limits the output noise and the output noise power is

\[
N_D = \int_{-B_D}^{B_D} N_0 df = 2N_0 B_D
\]

This case gives the detection gain

\[
\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 m^2 / 2N_0 B_D}{A_c^2 m^2 / 2N_0 B_T} = \frac{B_T}{B_D}
\]

**Problem 6.4**

This problem is identical to Problem 6.3 except that the predetection signal power is

\[
S_T = \frac{1}{2} A_c^2 \left[ 1 + a^2 \bar{m}_n^2 \right]
\]

and the postdetection signal power is

\[
S_D = A_c^2 a^2 \bar{m}_n^2
\]

The noise powers do not change.

Case I: \( B_D > \frac{1}{2}B_T \)

\[
\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 a^2 \bar{m}_n^2 / N_0 B_T}{A_c^2 \left[ 1 + a^2 \bar{m}_n^2 \right] / 2N_0 B_T} = \frac{2a^2 \bar{m}_n^2}{1 + a^2 \bar{m}_n^2}
\]

Case II: \( B_D < \frac{1}{2}B_T \)

\[
\frac{(SNR)_D}{(SNR)_T} = \frac{A_c^2 a^2 \bar{m}_n^2 / 2N_0 B_D}{A_c^2 \left[ 1 + a^2 \bar{m}_n^2 \right] / 2N_0 B_T} = \frac{a^2 \bar{m}_n^2}{1 + a^2 \bar{m}_n^2} = \frac{B_T}{B_D}
\]

**Problem 6.5**

Since the message signal is sinusoidal

\[
m_n(t) = \cos(8\pi t)
\]
Thus, 
\[ m_n^2 = 0.5 \]

The efficiency is therefore given by
\[ E_{ff} = \frac{(0.5)(0.8)^2}{1 + (0.5)(0.8)^2} = 0.2424 = 24.24\% \]

From (6.29), the detection gain is
\[ \frac{(SNR)_D}{(SNR)_T} = 2E_{ff} = 0.4848 = -3.14\text{dB} \]

and the output SNR is, from (6.33),
\[ (SNR)_D = 0.2424 - \frac{P_T}{N_0W} \]

Relative to baseband, the output SNR is
\[ \frac{(SNR)_D}{P_T/N_0W} = 0.2424 = -6.15\text{dB} \]

If the modulation index is increased to 0.9, the efficiency becomes
\[ E_{ff} = \frac{(0.5)(0.9)^2}{1 + (0.5)(0.9)^2} = 0.2883 = 28.83\% \]

This gives a detection gain of
\[ \frac{(SNR)_D}{(SNR)_T} = 2E_{ff} = 0.5765 = -2.39\text{dB} \]

The output SNR is
\[ (SNR)_D = 0.2883 - \frac{P_T}{N_0W} \]

which, relative to baseband, is
\[ \frac{(SNR)_D}{P_T/N_0W} = 0.2883 = -5.40\text{dB} \]

This represents an improvement of 0.75 dB.

**Problem 6.6**
The first step is to determine the value of $M$. First we compute

$$P \{ X > M \} = \int_M^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2} dy = Q \left( \frac{M}{\sigma} \right) = 0.005$$

This gives

$$\frac{M}{\sigma} = 2.57$$

Thus,

$$M = 2.57\sigma$$

and

$$m_n(t) = \frac{m(t)}{2.57\sigma}$$

Thus, since $\overline{m^2} = \sigma^2$

$$\overline{m_n^2} = \frac{2.57\sigma^2}{(2.57\sigma)^2} = \frac{\sigma^2}{6.605\sigma^2} = 0.151$$

Since $a = \frac{1}{2}$, we have

$$E_{ff} = \frac{0.151 \left( \frac{1}{2} \right)^2}{1 + 0.151 \left( \frac{1}{2} \right)^2} = 3.64\%$$

and the detection gain is

$$\frac{(SNR)_D}{(SNR)_T} = 2E = 0.0728$$

**Problem 6.7**

The output of the predetection filter is

$$e(t) = A_c [1 + am_n(t)] \cos [\omega_c t + \theta] + r_n(t) \cos [\omega_c t + \theta + \phi_n(t)]$$

The noise function $r_n(t)$ has a Rayleigh pdf. Thus

$$f_{r_n}(r_n) = \frac{r}{N} e^{-r^2/2N}$$

where $N$ is the predetection noise power. This gives

$$N = \frac{1}{2}n_c^2 + \frac{1}{2}n_s^2 = N_0 B_T$$

From the definition of threshold

$$0.99 = \int_0^{A_c} \frac{r}{N} e^{-r^2/2N} dr$$
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which gives
\[ 0.99 = 1 - e^{-A_c^2/2N} \]

Thus,
\[ \frac{A_c^2}{2N} = \ln(0.01) \]

which gives
\[ A_c^2 = 9.21 N \]

The predetection signal power is
\[ P_T = \frac{1}{2} A_c \left[ 1 + a^2 m_n^2 \right] \approx \frac{1}{2} A_c^2 [1 + 1] = A_c^2 \]

which gives
\[ \frac{P_T}{N} = \frac{A_c^2}{N} = 9.21 \approx 9.64 dB \]

Problem 6.8

Since \( m(t) \) is a sinusoid, \( m_n^2 = \frac{1}{2} \), and the efficiency is
\[ E_{ff} = \frac{1}{2} \frac{a^2}{1 + \frac{1}{2} a^2} = \frac{a^2}{2 + a^2} \]

and the output SNR is
\[ (SNR)_D = E_{ff} = \frac{a^2}{2 + a^2} \frac{P_T}{N_0 W} \]

In dB we have
\[ (SNR)_{D, dB} = 10 \log_{10} \left( \frac{a^2}{2 + a^2} \right) + 10 \log_{10} \frac{P_T}{N_0 W} \]

For \( a = 0.4 \),
\[ (SNR)_{D, dB} = 10 \log_{10} \frac{P_T}{N_0 W} - 11.3033 \]

For \( a = 0.5 \),
\[ (SNR)_{D, dB} = 10 \log_{10} \frac{P_T}{N_0 W} - 9.5424 \]

For \( a = 0.7 \),
\[ (SNR)_{D, dB} = 10 \log_{10} \frac{P_T}{N_0 W} - 7.0600 \]

For \( a = 0.9 \),
\[ (SNR)_{D, dB} = 10 \log_{10} \frac{P_T}{N_0 W} - 5.4022 \]
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The plot of $(SNR)_{dB}$ as a function of $P_T/N_0W$ in dB is linear having a slope of one. The modulation index only biases $(SNR)_{D_{dB}}$ down by an amount given by the last term in the preceding four expressions.

**Problem 6.9**

Let the predetection filter bandwidth be $B_T$ and let the postdetection filter bandwidth be $B_D$. The received signal (with noise) at the predetection filter output is represented

$$x_r(t) = A_c [1 + am_n(t)] \cos \omega_c t + n_c(t) \cos \omega_c t + n_s(t) \sin \omega_c t$$

which can be represented

$$x_r(t) = \{A_c [1 + am_n(t)] + n_c(t)\} \cos \omega_c t - n_s(t) \sin \omega_c t$$

The output of the square-law device is

$$y(t) = \{A_c [1 + am_n(t)] + n_c(t)\}^2 \cos^2 \omega_c t$$

Assuming that the postdetection filter removes the double frequency terms, $y_D(t)$ can be written

$$y_D(t) = \frac{1}{2} \{A_c [1 + am_n(t)] + n_c(t)\}^2 + \frac{1}{2} n_s^2(t)$$

Except for a factor of 2, this is (6.50). We now let $m_n(t) = \cos \omega_m t$ and expand. This gives

$$y_D(t) = \frac{1}{2} A_c^2 [1 + a \cos \omega_m t]^2 + A_c n_c(t)$$

$$+ A_c a n_c(t) \cos \omega_m t + \frac{1}{2} n_c^2(t) + \frac{1}{2} n_s^2(t)$$

We represent $y_D(t)$ by

$$y_D(t) = z_1(t) + z_2(t) + z_3(t) + z_4(t) + z_5(t)$$

where $z_i(t)$ is the $i^{th}$ term in the expression for $y_D(t)$. We now examine the power spectral density of each term. The first term is

$$z_1(t) = \frac{1}{2} A_c^2 \left(1 + \frac{1}{2} a^2\right) + A_c^2 a \cos \omega_m t + \frac{1}{4} A_c^2 a^2 \cos 2\omega_m t$$

$$z_2(t) = A_c n_c(t)$$

$$z_3(t) = A_c a n_c(t) \cos \omega_m t$$
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\[ z_4(t) = \frac{1}{2} n_c^2(t) \]

and

\[ z_5(t) = \frac{1}{2} n_s^2(t) \]

A plot of the PSD of each of these terms is illustrated in Figure 6.4. Adding these five terms together gives the PSD of \( y_D(t) \).

**Problem 6.10**

Assume sinusoidal modulation since sinusoidal modulation was assumed in the development of the square-law detector. For \( a = 1 \) and \( m_n(t) = \cos(2\pi f_m t) \) so that \( m_n^2 = 0.5 \), we have, for linear envelope detection (since \( P_T/N_0 W \gg 1 \), we use the coherent result),

\[
(SNR)_{D,\ell} = Ef = \frac{P_T}{2} \frac{a^2 m_n^2}{N_0 W} = \frac{1}{2} + \frac{1}{2} \frac{P_T}{N_0 W} = \frac{1}{3} \frac{P_T}{N_0 W}
\]

For square-law detection we have, from (6.61) with \( a = 1 \)

\[
(SNR)_{D,Sl} = 2 \left( \frac{a}{2 + a^2} \right)^2 \frac{P_T}{N_0 W} = 2 \frac{P_T}{9 N_0 W}
\]

Taking the ratio

\[
\frac{(SNR)_{D,Sl}}{(SNR)_{D,\ell}} = \frac{2}{\frac{1}{3} \frac{P_T}{N_0 W}} = \frac{2}{3} = -1.76 \text{ dB}
\]

This is approximately \(-1.8 \text{ dB}\).

**Problem 6.11**

For the circuit given

\[ H(f) = \frac{R}{R + j 2\pi f L} = \frac{1}{1 + j \left( \frac{2\pi f L}{R} \right)} \]

and

\[ |H(f)|^2 = \frac{1}{1 + \left( \frac{2\pi f L}{R} \right)^2} \]

The output noise power is

\[
N = \int_{-\infty}^{\infty} \frac{N_0}{2} \frac{df}{1 + \left( \frac{2\pi f L}{R} \right)^2} = N_0 \int_{0}^{\infty} \frac{1}{1 + x^2} \left( \frac{R}{2\pi L} \right) dx = \frac{N_0 R}{4L}
\]
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Figure 6.4:
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The output signal power is

\[ S = \frac{1}{2} \frac{A^2 R^2}{R^2 + (2\pi f_c L)^2} \]

This gives the signal-to-noise ratio

\[ \frac{S}{N} = \frac{2A^2 RL}{N_0 \left( R^2 + (2\pi f_c L)^2 \right)} \]

**Problem 6.12**

For the RC filter

\[ \frac{S}{N} = \frac{2A^2 RL}{N_0 \left( 1 + (2\pi f_c RC)^2 \right)} \]

**Problem 6.13**

The transfer function of the RC highpass filter is

\[ H(f) = \frac{R}{R + \frac{1}{j2\pi C}} = \frac{j2\pi f RC}{1 + j2\pi f RC} \]

so that

\[ H(f) = \frac{(j2\pi f RC)^2}{1 + (j2\pi f RC)^2} \]

Thus, The PSD at the output of the ideal lowpass filter is

\[ S_n(f) = \begin{cases} \frac{N_0}{2} \frac{(2\pi f RC)^2}{1 + (2\pi f RC)^2}, & |f| < W \\ 0, & |f| > W \end{cases} \]

The noise power at the output of the ideal lowpass filter is

\[ N = \int_{-W}^{W} S_n(f) \, df = N_0 \int_{0}^{W} \frac{(2\pi f RC)^2}{1 + (2\pi f RC)^2} \, df \]

with \( x = 2\pi f RC \), the preceding expression becomes

\[ N = \frac{N_0}{2\pi RC} \int_{0}^{2\pi RCW} \frac{x^2}{1 + x^2} \, dx \]

Since

\[ \frac{x^2}{1 + x^2} = 1 - \frac{1}{1 + x^2} \]
we can write
\[
N = \frac{N_0}{2\pi RC} \left\{ \int_{0}^{2\pi RCW} df - \int_{0}^{2\pi RCW} \frac{dx}{1 + x^2} \right\}
\]
or
\[
N = \frac{N_0}{2\pi RC} (2\pi RCW - \tan^{-1} (2\pi RCW))
\]
which is
\[
N = N_0 W - \frac{N_0 \tan^{-1} (2\pi RCW)}{2\pi RC}
\]
The output signal power is
\[
S = \frac{1}{2} A^2 |H(f_c)|^2 = \frac{A^2}{2} \frac{(2\pi f_c RC)^2}{1 + (2\pi f_c RC)^2}
\]
Thus, the signal-to-noise ratio is
\[
\frac{S}{N} = \frac{A^2}{2N_0} \frac{(2\pi f_c RC)^2}{1 + (2\pi f_c RC)^2} \frac{2\pi RC}{2\pi RCW - \tan^{-1} (2\pi RCW)}
\]
Note that as \(W \to \infty\), \(\frac{S}{N} \to 0\).

**Problem 6.14**
For the case in which the noise in the passband of the postdetection filter is negligible we have
\[
\bar{e}_Q^2 = \sigma_\phi^2, \quad \text{SSB and QDSB}
\]
and
\[
\bar{e}_D^2 = \frac{3}{4} \sigma_\phi^4, \quad \text{DSB}
\]
Note that for reasonable values of phase error variance, namely \(\sigma_\phi^2 \ll 1\), DSB is much less sensitive to phase errors in the demodulation carrier than SSB or QDSB. Another way of viewing this is to recognize that, for a given acceptable level of \(\bar{e}^2\), the phase error variance for can be much greater for DSB than for SSB or QDSB. The plots follow by simply plotting the two preceding expressions.

**Problem 6.15**
From the series expansions for \(\sin \phi\) and \(\cos \phi\) we can write
\[
\cos \phi = 1 - \frac{1}{2} \phi^2 + \frac{1}{24} \phi^4
\]
\[
\sin \phi = \phi - \frac{1}{6} \phi^3
\]
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Squaring these, and discarding all terms \( \phi^k \) for \( k > 4 \), yields

\[
\begin{align*}
\cos^2 \phi &= 1 - \phi^2 + \frac{1}{3} \phi^4 \\
\sin^2 \phi &= \phi^2 - \frac{1}{3} \phi^4
\end{align*}
\]

Using (6.70) and recognizing that \( m_1(t) \), \( m_2(t) \), and \( \phi(t) \) are independent, yields

\[
\bar{\varepsilon}^2 = \sigma^2_{m_1} - 2\sigma^2_{m_1} \cos \phi + \sigma^2_{m_1} \cos^2 \phi + \sigma^2_{m_2} \sin^2 \phi + \sigma_n^2
\]

Assuming \( \phi(t) \) to be a zero-mean process and recalling that \( \bar{\phi}_4 = 3\sigma_\phi^4 \) gives

\[
\begin{align*}
\bar{\cos}\phi &= 1 + \frac{1}{2} \sigma_\phi^2 + \frac{1}{8} \sigma_\phi^4 \\
\bar{\cos^2}\phi &= 1 - \sigma_\phi^2 + \sigma_\phi^4 \\
\bar{\sin^2}\phi &= \sigma_\phi^2 - \sigma_\phi^4
\end{align*}
\]

This yields

\[
\bar{\varepsilon}^2 = \sigma^2_{m_1} - 2\sigma^2_{m_1} \left( 1 + \frac{1}{2} \sigma_\phi^2 + \frac{1}{8} \sigma_\phi^4 \right) + \sigma^2_{m_1} (1 - \sigma_\phi^2 + \sigma_\phi^4) + \sigma^2_{m_2} (\sigma_\phi^2 - \sigma_\phi^4) + \sigma_n^2
\]

which can be expressed

\[
\bar{\varepsilon}^2 = \frac{3}{4} \sigma^2_{m_1} \sigma_\phi^4 + \sigma^2_{m_2} \sigma_\phi^2 - \sigma^2_{m_2} \sigma_\phi^4 + \sigma_n^2
\]

For QDSB we let \( \sigma^2_{m_1} = \sigma^2_{m_2} = \sigma_m^2 \). This gives

\[
\bar{\varepsilon}^2 = \sigma^2_{m_2} \left( \sigma_\phi^2 - \frac{1}{4} \sigma_\phi^4 \right) + \sigma_n^2
\]

For \( \sigma_\phi^2 >> \sigma_\phi^4 \), we have

\[
\bar{\varepsilon}^2 = \sigma_m^2 \sigma_\phi^2 + \sigma_n^2
\]

which yields (6.73). For DSB, we let \( \sigma^2_{m_1} = \sigma_m^2 \) and \( \sigma^2_{m_2} = 0 \). This gives (6.79)

\[
\bar{\varepsilon}^2 = \frac{3}{4} \sigma_m^2 \sigma_\phi^4 + \sigma_n^2
\]
Problem 6.16
From (6.73) and (6.79), we have

\[ 0.05 = \sigma_\phi^2 + \frac{\sigma_n^2}{\sigma_m^2}, \quad \text{SSB} \]

\[ 0.05 = \frac{3}{4} \left( \sigma_\phi^2 \right)^2 + \frac{\sigma_n^2}{\sigma_m^2}, \quad \text{DSB} \]

Thus we have the following signal-to-noise ratios

\[ \frac{\sigma_m^2}{\sigma_n^2} = \frac{1}{0.05 - \sigma_\phi^2}, \quad \text{SSB} \]

\[ \frac{\sigma_m^2}{\sigma_n^2} = \frac{1}{0.05 - \frac{3}{4} \left( \sigma_\phi^2 \right)^2}, \quad \text{DSB} \]

The SSB characteristic has an asymptote defined by

\[ \sigma_\phi^2 = 0.05 \]

and the DSB result has an asymptote defined by

\[ \frac{3}{4} \left( \sigma_\phi^2 \right)^2 = 0.05 \]

or

\[ \sigma_\phi^2 = 0.258 \]

The curves are shown in Figure 6.5. The appropriate operating regions are above and to the left of the curves. It is clear that DSB has the larger operating region.

Problem 6.17
Since we have a DSB system

\[ \overline{\epsilon_N^2} = \frac{3}{4} \sigma_\phi^4 + \frac{\sigma_n^2}{\sigma_m^2} \]

Let the bandwidth of the predetection filter be \( B_T \) and let the bandwidth of the pilot filter be

\[ B_p = \frac{B_T}{\alpha} \]

This gives

\[ \frac{\sigma_m^2}{\sigma_n^2} = \frac{N_0 B_T}{\sigma_m^2} = \frac{B_T}{B_p} \frac{N_0 B_p}{\sigma_m^2} = \frac{\alpha}{\rho} \]
From (6.85) and (6.86), we have
\[
\sigma^2_\phi = \frac{1}{k^2} \frac{1}{2\rho}
\]
so that
\[
\overline{\varepsilon_N^2} = \frac{3}{16} \frac{1}{k^4 \rho^2} + \frac{\alpha}{\rho}
\]
For an SNR of 15 dB
\[
\rho = 10^{1.5} = 31.623
\]
Using this value for \( \rho \) and with \( k = 4 \), we have
\[
\overline{\varepsilon_N^2} = 7.32 \left(10^{-7}\right) + 0.032\alpha
\]
The plot is obviously linear in \( \alpha \) with a slope of 0.032. The bias, \( 7.32 \times 10^{-7} \), is negligible. Note that for \( k > 1 \) and reasonable values of the pilot signal-to-noise ratio, \( \rho \), the first term (the bias), which arises from pilot phase jitter, is negligible. The performance is limited by the additive noise.

**Problem 6.18**
The mean-square error
\[
\overline{\varepsilon^2 (A, \tau)} = E \{ [y(t) - Ax(t - \tau)]^2 \}
can be written
\[ \varepsilon^2(A, \tau) = E \{ y^2(t) - 2Ax(t-\tau)y(t) + A^2x^2(t-\tau) \} \]

In terms of autocorrelation and cross correlation functions, the mean-square error is
\[ \varepsilon^2(A, \tau) = R_y(0) - 2AR_{xy}(\tau) + A^2R_x(0) \]
Letting \( P_y = R_y(0) \) and \( P_x = R_x(0) \) gives
\[ \varepsilon^2(A, \tau) = P_y - 2AR_{xy}(\tau) + A^2P_x \]
In order to minimize \( \varepsilon^2 \) we choose \( \tau = \tau_m \) such that \( R_{xy}(\tau) \) is maximized. This follows since the crosscorrelation term is negative in the expression for \( \varepsilon^2 \). Therefore,
\[ \varepsilon^2(A, \tau_m) = P_y - 2AR_{xy}(\tau_m) + A^2P_x \]
The gain \( A_m \) is determined from
\[ \frac{d\varepsilon^2}{dA} = -2AR_{xy}(\tau_m) + A^2P_x = 0 \]
which yields
\[ A_m = \frac{R_{xy}(\tau_m)}{P_x} \]
This gives the mean-square error
\[ \varepsilon^2(A_m, \tau_m) = P_y - 2\frac{R_{xy}^2(\tau_m)}{P_x} + \frac{R_{xy}^2(\tau_m)}{P_x} \]
which is
\[ \varepsilon^2(A_m, \tau_m) = P_y - \frac{P_{xy}^2(\tau_m)}{P_x} \]
The output signal power is
\[ S_D = E \{ [A_m x(t-\tau_m)]^2 \} = A_m^2P_x \]
which is
\[ S_D = \frac{R_{xy}^2(\tau_m)}{P_x} = \frac{R_{xy}^2(\tau_m)}{R_x(0)} \]
Since \( N_D \) is the error \( \varepsilon^2(A_m, \tau_m) \) we have
\[ \frac{S_D}{N_D} = \frac{R_{xy}^2(\tau_m)}{R_x(0)R_y(0) - R_{xy}^2(\tau_m)} \]
Note: The gain and delay of a linear system is often defined as the magnitude of the transfer and the slope of the phase characteristic, respectively. The definition of gain and delay suggested in this problem is useful when the magnitude response is not linear over the frequency range of interest.

**Problem 6.19**
The single-sided spectrum of a stereophonic FM signal and the noise spectrum is shown in Figure 6.6. The two-sided noise spectrum is given by

\[ S_nF(f) = \frac{K_D^2}{A_C^2} N_0 f, \quad -\infty < f < \infty \]

The predetection noise powers are easily computed. For the \(L + R\) channel

\[ P_{n,L+R} = 2 \int_{0}^{15,000} \frac{K_D^2}{A_C^2} N_0 f^2 df = 2.25 \times 10^{12} \frac{K_D^2}{A_C^2} N_0 \]

For the \(L - R\) channel

\[ P_{n,L-R} = 2 \int_{23,000}^{53,000} \frac{K_D^2}{A_C^2} N_0 f^2 df = 91.14 \times 10^{12} \frac{K_D^2}{A_C^2} N_0 \]

Thus, the noise power on the \(L - R\) channel is over 40 times the noise power in the \(L + R\) channel. After demodulation, the difference will be a factor of 20 because of 3 dB detection gain of coherent demodulation. Thus, the main source of noise in a stereophonic system is the \(L - R\) channel. Therefore, in high noise environments, monophonic broadcasting is preferred over stereophonic broadcasting.

**Problem 6.20**
The received FDM spectrum is shown in Figure 6.7. The \(k\)th channel signal is given by

\[ x_k(t) = A_k m_k(t) \cos 2\pi kf_1 t \]

Since the guardbands have spectral width \(4W\), \(f_k = 6kW\) and the \(k\)th channel occupies the frequency band

\[ (6k - 1)W \leq f \leq (6k + 1)W \]

Since the noise spectrum is given by

\[ S_{nF} = \frac{K_D^2}{A_C^2} N_0 f^2, \quad |f| < B_T \]

The noise power in the \(k\)th channel is

\[ N_k = B \int_{(6k-1)W}^{(6k+1)W} f^2 df = \frac{BW^3}{3} \left[ (6k + 1)^3 - (6k - 1)^3 \right] \]
where $B$ is a constant. This gives

$$N_k = \frac{BW^3}{3} (216k^2 + 2) \approx 72BW^3k^2$$

The signal power is proportional to $A_k^2$. Thus, the signal-to-noise ratio can be expressed as

$$(SNR)_D = \frac{\lambda A_k^2}{k^2} = \lambda \left( \frac{A_k}{k} \right)^2$$

where $\lambda$ is a constant of proportionality and is a function of $K_D, W, A_C, N_0$. If all channels are to have equal $(SNR)_D$, $A_k/k$ must be a constant, which means that $A_k$ is a linear function of $k$. Thus, if $A_1$ is known, $A_k = k A_1, k = 1, 2, ..., 7$. $A_1$ is fixed by setting the $SNR$ of Channel 1 equal to the $SNR$ of the unmodulated channel. The noise power of the unmodulated channel is

$$N_0 = B \int_0^W f^2 df = \frac{B}{3} W^3$$

yielding the signal-to-noise ratio

$$(SNR)_D = \frac{P_0}{\frac{B}{3} W^3}$$

where $P_0$ is the signal power.

**Problem 6.21**
From (6.132) and (6.119), the required ratio is

\[ R = \frac{2 \frac{K^2}{\Delta f} N_0 f_3^3 \left( \frac{W}{f_3} - \tan^{-1} \frac{W}{f_3} \right)}{3 \frac{K^2}{\Delta f} N_0 W^3} \]

or

\[ R = 3 \left( \frac{f_3}{W} \right)^3 \left( \frac{W}{f_3} - \tan^{-1} \frac{W}{f_3} \right) \]

This is shown in Figure 6.8.

For \( f_3 = 2.1 kHz \) and \( W = 15 kHz \), the value of \( R \) is

\[ R = 3 \left( \frac{2.1}{15} \right)^2 \left( \frac{15}{2.1} - \tan^{-1} \frac{15}{2.1} \right) = 0.047 \]

Expressed in dB this is

\[ R = 10 \log_{10}(0.047) = -13.3 \text{ dB} \]

The improvement resulting from the use of preemphasis and deemphasis is therefore 13.3 dB. Neglecting the \( \tan^{-1}(W/f_3) \) gives an improvement of \( 21 - 8.75 = 12.25 \text{ dB} \). The difference is approximately 1 dB.

**Problem 6.22**

From the plot of the signal spectrum it is clear that

\[ k = \frac{A}{W^2} \]
Thus the signal power is

\[ S = 2 \int_0^W \frac{A}{W^2} f^2 df = \frac{2}{3} AW \]

The noise power is \( N_0 B \). This yields the signal-to-noise ratio

\[ (SNR)_1 = \frac{2}{3} \frac{AW}{N_0 B} \]

If \( B \) is reduced to \( W \), the SNR becomes

\[ (SNR)_2 = \frac{2}{3} \frac{A}{N_0} \]

This increases the signal-to-noise ratio by a factor of \( B/W \).

**Problem 6.23**

From the definition of the signal we have

\[
x(t) = A \cos 2\pi f_c t \\
\frac{dx}{dt} = -2\pi f_c A \sin 2\pi f_c t \\
\frac{d^2x}{dt^2} = -(2\pi f_c)^2 A \cos 2\pi f_c t
\]
The signal component of $y(t)$ therefore has power

$$S_D = \frac{1}{2} A^2 (2\pi f_c)^4 = 8A^2\pi^4 f_c^4$$

The noise power spectral density at $y(t)$ is

$$S_n(f) = \frac{N_0}{2} (2\pi f)^4$$

so that the noise power is

$$N_D = \frac{N_0}{2} (2\pi)^4 \int_{-W}^{W} f^4 df = \frac{16}{5} N_0\pi^4 W^5$$

Thus, the signal-to-noise ratio at $y(t)$ is

$$(SNR)_D = 5 \frac{A^2}{2 N_0 W} \left( \frac{f_c}{W} \right)^4$$

**Problem 6.24**

Since the signal is integrated twice and then differentiated twice, the output signal is equal to the input signal. The output signal is

$$y_s(t) = A \cos 2\pi f_c t$$

and the output signal power is

$$S_D = \frac{1}{2} A^2$$

The noise power is the same as in the previous problem, thus

$$N_D = \frac{16}{5} N_0\pi^4 W^5$$

This gives the signal-to-noise ratio

$$(SNR)_D = \frac{5}{32\pi^4} \frac{A^2}{N_0 W^5}$$

**Problem 6.25**

The signal-to-noise ratio at $y(t)$ is

$$(SNR)_D = \frac{2A}{N_0} \left( \frac{f_3}{W} \right) \tan^{-1} \frac{W}{f_3}$$
The normalized $(SNR)_D$, defined by $(SNR)_D/(2A?N_0)$, is illustrated in Figure 6.9.

**Problem 6.26**
The instantaneous frequency deviation in Hz is

$$\delta f = f_{dm}(t) = x(t)$$

where $x(t)$ is a zero-mean Gaussian process with variance $\sigma_x^2 = f_d^2\sigma_m^2$. Thus,

$$|\delta f| = |x| = \int_{-\infty}^{\infty} \frac{|x|}{\sqrt{2\pi}\sigma_x} e^{-x^2/2\sigma_x^2} dx$$

Recognizing that the integrand is even gives

$$|\delta f| = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_x} \int_0^{\infty} x e^{-x^2/2\sigma_x^2} dx = \sqrt{\frac{2}{\pi}} \sigma_x$$

Therefore,

$$|\delta f| = \sqrt{\frac{2}{\pi}} f_d\sigma_m$$
6.1. PROBLEMS

Substitution into (6.148) gives

\[(SNR)_D = \frac{3 \left( \frac{f_d}{W} \right)^2 m_2^2 P_r}{1 + 2\sqrt{3} \frac{B_r}{W} Q \left[ \frac{A_r}{\sqrt{N_0 B_r}} \right] + 6 \sqrt{\frac{2L \sigma_m}{W}} \exp \left[ - \frac{A_r^2}{2N_0 B_r} \right]} \]

The preceding expression can be placed in terms of the deviation ratio, \(D\), by letting

\[D = \frac{f_d}{W}\]

and

\[B_T = 2(D + 1)W\]

**Problem 6.27**

(Note: This problem was changed after the first printing of the book. The new problem, along with the solution follow.)

Assume a PCM system in which the the bit error probability \(P_b\) is sufficiently small to justify the approximation that the word error probability is \(P_w \approx nP_b\). Also assume that the threshold value of the signal-to-noise ratio, defined by (6.175), occurs when the two denominator terms are equal, i.e., the effect of quantizing errors and word errors are equivalent. Using this assumption derive the threshold value of \(P_T/N_0 B_p\) in dB for \(n = 4, 8,\) and 12. Compare the results with Figure 6.22 derived in Example 6.5.

With \(P_w = nP_b\), equating the two terms in the denominator of (6.175) gives

\[2^{-2n} = nP_b(1 - 2^{-2n})\]

Solving for \(P_b\) we have

\[P_b = \frac{1}{n} \cdot \frac{2^{-2n}}{1 - 2^{-2n}} \approx \frac{1}{n} \cdot 2^{-2n}\]

From (6.178)

\[\exp \left( - \frac{P_T}{2N_0 B_p} \right) = 2P_b \approx \frac{1}{n} 2^{(1 - 2n)}\]

Solving for \(P_T/N_0 B_p\) gives, at threshold,

\[\frac{P_T}{N_0 B_p} \approx -2 \ln \left( \frac{1}{n} 2^{(1 - 2n)} \right) = 2(2n - 1)\ln(2) + 2\ln(2)\]

The values of \(P_T/N_0 B_p\) for \(n = 4, 8,\) and 12 are given in Table 6.1. Comparison with Figure 6.22 shows close agreement.
### Table 6.1:

<table>
<thead>
<tr>
<th>$n$</th>
<th>Threshold value of $P_T/N_0B_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>10.96 dB</td>
</tr>
<tr>
<td>8</td>
<td>13.97 dB</td>
</tr>
<tr>
<td>12</td>
<td>15.66 dB</td>
</tr>
</tbody>
</table>

#### Problem 6.28

The peak value of the signal is 7.5 so that the signal power is

$$S_D = \frac{1}{2} (7.5)^2 = 28.125$$

If the A/D converter has a wordlength $n$, there are $2^n$ quantizing levels. Since these span the peak-to-peak signal range, the width of each quantization level is

$$S_D = \frac{15}{2^n} = 15 \left(2^{-n}\right)$$

This gives

$$\bar{\varepsilon}^2 = \frac{1}{12}S_D^2 = \frac{1}{12} (15)^2 \left(2^{-2n}\right) = 18.75 \left(2^{-2n}\right)$$

This results in the signal-to-noise ratio

$$SNR = \frac{S_D}{\bar{\varepsilon}^2} = \frac{28.125}{18.75} \left(2^{2n}\right) = 1.5 \left(2^{2n}\right)$$